

Entanglement Swapping of Generalized Cat States and Secret Sharing

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Abstract

We introduce generalized cat states for d -level systems and obtain concise formulas for their entanglement swapping with generalized Bell states. We then use this to provide both a generalization to the d -level case and a transparent proof of validity for an already proposed protocol of secret sharing based on entanglement swapping.

Keywords: d -Level States, GHZ States, Entanglement Swapping, Secret Sharing.

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1 Introduction

There are numerous uses of spatially separated entangled pairs of particles such as quantum key distribution and secret sharing [1, 2, 3, 4, 5], teleportation [6], superdense coding [7], and cheating bit commitment [8]. It has been argued that three or more spatially separated particles in an entangled state (such as a GHZ or cat state [9]) may have similar or even broader applications. It is first essential to distinguish between GHZ states and cat states. By an n -party cat state, we mean a highly entangled state of n particles, while by a GHZ state we mean one, which contradicts an interpretation in terms of any local hidden variable theory. Constructing the latter types of states for general multi-level multi-particle systems is quite a difficult task, although some general criteria have been outlined for their identification [10]. On the other hand we will show that one can easily define n -party cat states with nice properties (i.e: entanglement swapping) which allows them to be used in a secret sharing protocol and possibly in many other communication protocols, although they may not be used for testing non-locality properties of quantum mechanics. More recently, applications such as reducing communication complexity [11] quantum telecomputation [12], and networked cryptographic conferencing [13, 14] have also been suggested as possible new applications of these multi-particle entangled states.

For practical applications such as mentioned above, there has been much interest in manipulating entangled states of many particles [15, 16, 17, 18]. In particular it has been shown that by appropriate Bell measurements, entanglement can be swapped between different particles [15], a scheme which has been generalized to multi particle case in [17]. In fact to the question of “Which particles get entangled when we make cat state measurement on a group of particles?”, there is a general pencil and paper rule which provides the answer [17]. One just has to connect the particles being measured to frame a polygon and those not being measured to frame a complementary polygon. These two polygons represent the two multi-particle cat states obtained after the manipulation. However for most applications it is highly necessary to know exactly the type (e.g. the labels) of the cat state that the particles are forming and a knowledge of only the particles sharing the entanglement is not enough. In fact in almost any of the communication protocols mentioned above, the information to be transferred is encoded in the type of the labels of the cat states involved. For this reason one needs also a simple pencil and paper rule for determining the types and the labels of the cat states involved in a swapping process.

It may not be so illuminating to derive a general formula for such a purpose, although it is rather straightforward to do so. However if we restrict to the most common type of swapping, that is, the swapping of a cat state and a Bell state, then transparent, graphical and very useful rules can be derived as we will show below. Furthermore we will derive the rules for general d -level systems. We will then apply these rules to the quantum key distribution and secret sharing protocols of [3, 4] and show that the rules of encoding and decoding of this protocol, expressed otherwise only in tables, even when few parties are involved [4] can be neatly expressed by closed formulas in the general case.

The structure of this paper is as follows. In section 2 we review the basic properties

of d -level Bell states [19] and introduce d -level cat states. In section 3 we derive simple graphical rules for entanglement swapping of d -level Bell and cat states. We then apply in section 4, these rules to the secret sharing protocol of Cabello [4] to see how simple the encoding and decoding rules of this protocol are. We conclude the paper with a discussion.

2 Generalized cat states for d -level systems

In studying d -level states and their entanglement properties we are following an interesting trend to generalize the well known quantum algorithms and protocols of quantum computation and communication to non-binary systems, like quantum gates for qudits [20], quantum error correcting codes [21, 22], and generalization of the BB84 protocol [23] for quantum key distribution[2]. (For a review on quantum key distribution see [24].)

In fact considerations of quantum hardware may bring about some advantage to non-binary systems, since bigger Hilbert spaces can be made by coupling fewer d -dimensional systems than 2-dimensional ones, and it is well known that complete coupling of quantum bits gets much more difficult with the number of qubits increasing. Some researchers have even considered quantum computation and communication with continuous variables [25, 26]. Besides these, it is very instructive to study quantum computation and communication for d -level systems (qudits) to understand them in a general dimension-free setting.

We start by reviewing a generalization of the familiar Bell states for qudits introduced in [19]. These are a set of d^2 maximally entangled states which form an orthonormal basis for the space of two qudits. Their explicit forms are:

$$|\Psi(u_1, u_2)\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \zeta^{ju_1} |j, j + u_2\rangle \quad (1)$$

where $\zeta = e^{\frac{2\pi i}{d}}$ and u_1 and u_2 run from 0 to $d-1$. Each Bell state is thus characterized by a pair of two Z_d labels. For $d = 2$ these states reduce to the familiar Bell states, usually denoted by $|\Psi^\pm\rangle$ and $|\Phi^\pm\rangle$. One can also expand any computational basis vector in terms of Bell states:

$$|j, k\rangle = \frac{1}{\sqrt{d}} \sum_{u=0}^{d-1} \zeta^{-ju} |\Psi(u, k - j)\rangle. \quad (2)$$

It is also useful to consider a generalization of the familiar Hadamard gate to the d -level case. It is defined [27, 28] as follows:

$$H = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} \zeta^{ij} |i\rangle\langle j|. \quad (3)$$

This operator is really not new and it is known as the quantum fourier transform when $d = 2^n$. In that case it acts on n qubits. Here we are assuming it to be a basic

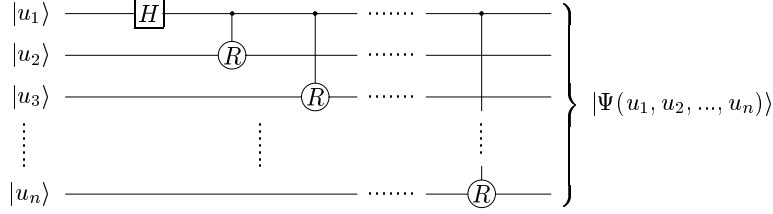


Figure 1: Circuit for constructing d -level cat states

gate on one single qudit, in the same way that the ordinary Hadamard gate is a basic gate on one qubit. It is also useful to generalize the NOT and the CNOT gates. We note that in the context of qubits, the NOT gate, is basically a mod-2 adder. For qudits this operator gives way to a mod- d adder, or a Right-Shift gate.

$$R|j\rangle = |j + 1\rangle \quad \text{mod } d, \quad (4)$$

where here and hereafter all our additions are defined mod d . Note that $R^d = I$, compared to $NOT^2 = I$. For any unitary operator U , the controlled operator U_c is naturally generalized as follows:

$$U_c(|i\rangle \otimes |j\rangle) = |i\rangle \otimes U^i|j\rangle \quad (5)$$

Here the first and the second qudits are respectively the controller and the target qudits. In particular the controlled shift gates which play the role of CNOT gate, act as follows:

$$R_c|i, j\rangle = |i, j + i\rangle \quad (6)$$

Equipped with the d -level Hadamard and CNOT (R_c) gates, one can construct d -level cat states simply as in the 2-level case by the circuit shown in fig. (1), where $|u_1, u_2, \dots, u_n\rangle$ is a computational basis vector, in which $u_i \in \{0, 1, 2, \dots, d-1\}$. The resulting cat state will be:

$$|\Psi(u_1, u_2, \dots, u_n)\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \zeta^{ju_1} |j, j + u_2, j + u_3, \dots, j + u_n\rangle. \quad (7)$$

These states are orthonormal, $\langle \Psi(v_1, \dots, v_n) | \Psi(u_1, \dots, u_n) \rangle = \delta_{u_1, v_1} \dots \delta_{u_n, v_n}$ and complete: any computational basis vector can be expanded in terms of these generalized cat states:

$$|u_1, u_2, u_3, \dots, u_n\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \zeta^{-ju_1} |\Psi(j, u_2 - u_1, u_3 - u_1, \dots, u_n - u_1)\rangle, \quad (8)$$

Quite analogously to the 2-level case, one can generate a cat state of n particles from a cat state of $n - 1$ particles in two ways, either using Zeilinger et. al. method [18],

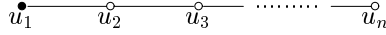


Figure 2: Visualization of $|\Psi(u_1, u_2, \dots, u_n)\rangle$ cat state

that is: acting by an R_c gate on one particle of the $(n-1)$ -cat state and one qudit of a Bell state, subsequently measuring the target qudit, or by using the method of [17] by performing a Bell state measurement on two particles, one from an $(n-1)$ -cat state and the other from a 3-cat state, projecting the rest onto an n -cat state.

3 Some Simple Rules for Entanglement Swapping

Entanglement swapping is nothing but tensor multiplying two cat states, expanding them in the computational basis of the product space, swapping a subset of particles and then re-expanding the resulting state in terms of the new cat states. The idea and the essential calculation is best illustrated by the simplest example, that is, swapping two Bell states. Suppose particles 1 and 2 are in a Bell state $|\Psi(u_1, u_2)\rangle_{1,2}$ and particles 3 and 4 are in a Bell state $|\Psi(v_1, v_2)\rangle_{3,4}$. This state of the four particles is equal to:

$$\begin{aligned}
& \frac{1}{d} \sum_{j,j'} \zeta^{ju_1+j'v_1} |j, j+u_2\rangle_{1,2} |j', j'+v_2\rangle_{3,4} \\
&= \frac{1}{d} \sum_{j,j'} \zeta^{ju_1+j'v_1} |j, j'+v_2\rangle_{1,4} |j', j+u_2\rangle_{3,2} \\
&= \frac{1}{d^2} \sum_{j,j',w,w'} \zeta^{ju_1+j'v_1} \zeta^{-jw-j'w'} |\Psi(w, j'+v_2-j)\rangle_{1,4} |\Psi(w', j+u_2-j')\rangle_{3,2} \quad (9)
\end{aligned}$$

Changing the variables ($j'-j \rightarrow \ell$), and using the identity $\frac{1}{d} \sum_{j=0}^{d-1} \zeta^{jn} = \delta(n, 0)$, and rearranging terms we finally arrive at:

$$|\Psi(u_1, u_2)\rangle_{1,2} |\Psi(v_1, v_2)\rangle_{3,4} = \frac{1}{d} \sum_{k,\ell} \zeta^{-k\ell} |\Psi(u_1+k, v_2+\ell)\rangle_{1,4} |\Psi(v_1-k, u_2-\ell)\rangle_{3,2} \quad (10)$$

It is customary to represent a cat state by a polygon. However a cat state is not symmetric and a polygon can not represent it properly. In fact as it is clear from 7 that a cat state is symmetric under the interchange of both the labels and the particles from 2 to n , i.e:

$$|\Psi(u_1, \dots, u_k, \dots, u_l, \dots, u_n)\rangle_{1,\dots,k,\dots,l,\dots,n} = |\Psi(u_1, \dots, u_l, \dots, u_k, \dots, u_n)\rangle_{1,\dots,l,\dots,k,\dots,n} \quad (11)$$

however it has no such symmetry under the interchange of the first particle with another one. We therefore depict a cat state by a line with n nodes on it, distinguishing the first node from the others by by assigning a black circle to it compared with empty circles assigned to others (fig. (2)). With this convention, the result of swapping

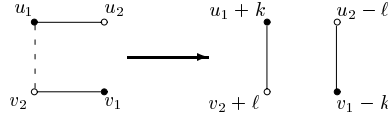


Figure 3: Entanglement swapping of d -level Bell states

calculated in equation 9 can be depicted as in fig. (3), where we have ignored the coefficients of the expansion and the arrow is meant to imply that the right hand side is a possible outcome of the Bell measurement performed on the left hand side particles designated by dashed line. The simple rule is that the sum of labels on the black nodes and white nodes are conserved separately in such a swapping. We will see that this type of rule will also hold true with slight modifications in swapping of Bell states and cat states.

We now derive formulas for swapping Bell states and cat states. We distinguish two cases, one in which a Bell state measurement involves the first particle (the black node) of the cat state and one in which it does not. For the first case we find after some straightforward calculations:

$$|\Psi(u_1, u_2, \dots, u_n)\rangle_{1,2,\dots,n} \otimes |\Psi(v, v')\rangle_{s,s'} = \frac{1}{d} \sum_{k,\ell} \zeta^{-\ell k} |\Psi(v+k, u_2-\ell, u_3-\ell, \dots, u_n-\ell)\rangle_{s,2,3,4,\dots,n} \otimes |\Psi(u_1-k, v'+\ell)\rangle_{1,s'} \quad (12)$$

This formula is depicted graphically in fig. (4-a). Again we see a simple rule in terms of the conservation of the labels on the black and white nodes. For the second case where the Bell state measurement does not involve the black node of the cat state we find:

$$|\Psi(u_1, u_2, \dots, u_n)\rangle_{1,2,3,\dots,n} \otimes |\Psi(v, v')\rangle_{s,s'} = \frac{1}{d} \sum_{k,\ell} \zeta^{-\ell k} |\Psi(u_1+k, u_2, u_3, \dots, v'+\ell, \dots, u_n)\rangle_{1,2,\dots,s',\dots,n} \otimes |\Psi(v-k, u_m-\ell)\rangle_{s,m} \quad (13)$$

This is depicted in fig. (4-b).

4 Secret key sharing by entanglement swapping

Among the many applications of entanglement swapping mentioned in the introduction, in this section we consider the secret key sharing protocol proposed by Cabello in [3, 4]. In this protocol n members of a group want to agree upon a secret key (For $n = 2$, we have the simple QKD scheme). The key is to be such that no proper subset of the group can determine it and its determination requires the cooperation of all members of the group. In the protocol proposed by Cabello [4] the n members of the group share an n -cat state, and each of them has also a Bell state. Each of the members swaps her or his Bell state with the cat state, and then all of them send

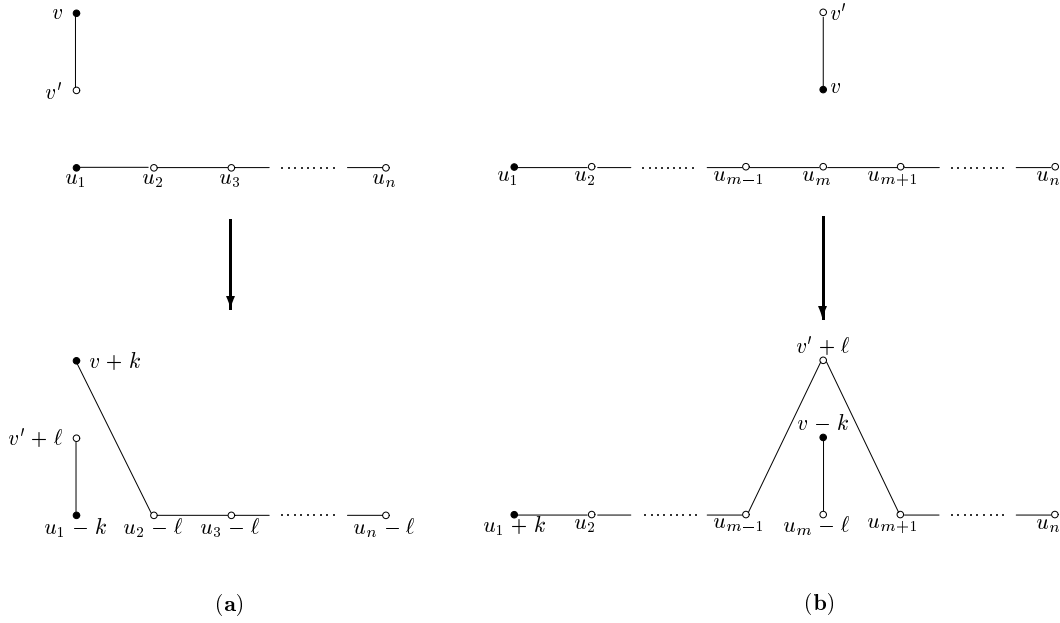


Figure 4: Entanglement swapping between a cat state and a Bell state

the resulting cat state to one of the members say Alice, who measures the cat state and announces the result of her measurement in public. It is then argued that by using this knowledge and the result of their own Bell measurements, the members of the group can all determine the result of the Bell measurement of Alice, which is to act as a random two bit key. In [4], it is shown by way of a couple of examples for 3 and 4 parties and compiling the results of measurements in tables, that this is indeed possible. Here we generalize the results of [4] in two respects. First we consider general d -level systems instead of two level ones, second by using our simple rules for entanglement swapping we derive general and concise formulas for determining the final secret key in terms of measurements of individuals members. As mentioned above, we can carry out all of the analysis graphically, where by our graphics we not only imply the particles which get entangled under swapping but also indicate precisely the entangled states they form in this process. The first stage of the process is depicted in fig. (5-a), where each member (i), has a Bell state (v_i, v'_i) and all the members share also a cat state (u_1, u_2, \dots, u_n) . When the first member whom we call Alice, performs her Bell measurement the entanglement swaps to the form shown in fig. (5-b), where we have used the first rule of fig. (4-a). Subsequently members numbered 2, 3, ... and n perform their Bell measurement and the states swap to that of fig. (5-c). The random two dit key is the pair of labels of Alice's Bell state, that is $(u_1 - k_1, v'_1 + \ell_1)$. At this stage the cat state is sent to Alice, she measures the state and announces the labels $(v_1 + k_1 + k_2 + \dots + k_n, v'_2 + \ell_2, v'_3 + \ell_3, \dots, v'_n + \ell_n)$ of this state in public. It is now clear that each member of the group say the i -th one, ($i = 2, 3, \dots, n$), knowing his own Bell state (v_i, v'_i) at the beginning of the pro-

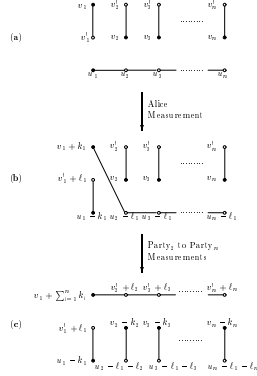


Figure 5: A protocol for d -level secret sharing

tol, his final Bell state $(v_i - k_i, u_i - \ell_1 - \ell_i)$ and the publicly announced cat state, can independently determine l_1 and hence the second label of the secret key, $v'_1 + l_1$. (Note that the shared cat state labels and all the Bell labels including those of Alice (v_1, v'_1) are assumed to be known to all the members at the beginning of the protocol). However to determine the first label of the key, that is $u_1 - k_1$, the members need a knowledge of k_1 , which no subset of the group can determine independently. It can only be found by sharing their values of $k_i, i = 2, 3, \dots, n$ with each other. Once this is done all members can determine the value of k_1 from the publicly announced label of Alice $v_1 + k_1 + k_2 + \dots + k_n$. The way we have presented this protocol, which starts with general Cat and Bell states, rather than with special ones with say all the labels being zero (i.e. $\Phi(0, 0)\rangle$), has the advantage that it shows how the encoding and decoding scheme works for consecutive qudits, when the same Bell and Cat states are re-used. To compare our results with those of [4], it is enough to set all the original labels $u_i, v_i, v'_i = 0$. It is then easy to see from fig. (5) that our results completely match the tables presented in that article.

5 Discussion

We have provided closed formulas for entanglement swapping of d -level cat states and Bell states. We have then used our formulas for providing transparent proof for the validity of a secret sharing protocol between n parties based on entanglement swapping. We expect that our graphical method of representing cat states and our formulas for entanglement swapping (ES) may find applications in every ES-based protocol in quantum communication.

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